Field quantization in a plasma: Photon mass and charge

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It is shown here that a straightforward procedure can be used to quantize the linearized equations for an electromagnetic field in a plasma. This leads to a definition of an effective mass for the transverse photons, and a different one for the longitudinal photons, or plasmons. Both masses are simply proportional to the electron plasma density. A nonlinear perturbative analysis can also be used to extend the quantization procedure, in order to include the ponderomotive force effects. This leads to the definition of a photon charge operator. The mean value of this operator, for a quantum state with a photon occupation number equal to 1, is the equivalent charge of the photon in a plasma.

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The concepts of effective mass and equivalent charge of a photon in a plasma were recently introduced in the context of photon dynamics in nonstationary plasmas. The photon effective mass is a linear concept, which is intimately related to the possibility of accelerating photons by moving plasma perturbations, such as an ionization front or a wakefield produced by intense and short laser pulses propagating in a neutral gas [1,2].

The photon equivalent charge, in contrast, is a nonlinear concept, intimately related with the existence of the ponderomotive force effect [3,4]. It allows for the reaction of radiation onto the plasma electrons, and can lead to the possibility of photon Landau damping of electron plasma waves [5].

One question is often raised as to the ultimate nature of these two concepts. In particular, we may question if they are not just artifacts of a classical statistical description of the electromagnetic radiation in a plasma or if, in contrast, they can be described by more fundamental equations in the context of the quantum theory of radiation. This question is addressed in the present work, where we deal with the process of electromagnetic field quantization in a plasma, and establish the quantum definition of the above two concepts.

We consider high-frequency fields and, for that reason, we completely neglect the ion dynamics. Furthermore, we assume an infinite, homogeneous, and isotropic plasma.

We start from Maxwell's equations, and from nonrelativistic electron equations of motion. From these basic equations, it is an easy matter to derive the propagation equations for the scalar and vector potentials ψ and $\vec{A}$ in the following form:

$$\nabla^2\psi = \frac{e}{\epsilon_0}\tilde{n}, \quad \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0 \vec{J}_\perp . \tag{1}$$

Here $\tilde{n}=(n-n_0)$ is the electron density perturbation with respect to the equilibrium value n_0, and $\vec{J}_\perp$ is the transverse part of the electric current, defined by $\nabla\cdot\vec{J}_\perp=0$. We are using the Coulomb gauge, which means that $\nabla\cdot\vec{A}=0$.

If we linearize the electron equations of motion, we can also obtain

$$\vec{J}_\perp = -en_0\vec{v}_\perp, \quad \vec{v}_\perp = \frac{e}{m}\vec{A} \tag{2}$$

and

$$\left(\frac{\partial^2}{\partial t^2} - S_e^2\nabla^2\right)\tilde{n} = \frac{en_0}{m}\nabla^2\psi \tag{3}$$

with $S_e^2=3v_e^2$, where $v_e=\sqrt{T/m}$ is the electron thermal velocity.

Replacing Eqs. (2) and (3) in Eq. (1), we obtain the linear propagation equations for the scalar and vector potentials in a plasma:

$$\left(\nabla^2 - \frac{1}{S_e^2}\frac{\partial^2}{\partial t^2}\right)\psi = \frac{\omega_p^2}{S_e^2}\psi \tag{4}$$

and

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = \frac{\omega_p^2}{c^2}\vec{A}. \tag{5}$$

We see that these two equations are formally identical, apart from the important difference between the electron thermal velocity S_e and the velocity of light in vacuum c. If the equality $S_e=c$ was verified, these two equations would be those of a massive vector field. This analogy with a massive vector field was already noted by Anderson in 1957 [6], and inspired the theory of the massive Higgs boson [7].

Following the usual quantization procedure, we can easily establish an energy operator, of the form

$$W = \sum_{\lambda=1}^{3}\int w(k,\lambda)\frac{d\vec{k}^3}{(2\pi)^3} \tag{6}$$

where the energy density operator $w(k,\lambda)$ is defined as

$$w(k,\lambda) = \hbar\omega_k(\lambda)[a^\dagger(k,\lambda)a(k,\lambda)+1/2]. \tag{7}$$

Here $a^\dagger(k,\lambda)$ and $a(k,\lambda)$ are the creation and destruction operators for the three available photon states $\lambda=1$, 2, and 3, with wave vector $\vec{k}$, and the corresponding photon frequencies are determined by the dispersion relation

$$\omega_k(\lambda) = \sqrt{k^2 c_\lambda^2 + \omega_p^2}, \tag{8}$$

where $c_\lambda = c$ for $\lambda = 1$ and 2, and $c_\lambda = S_e$ for $\lambda = 3$. Obviously, the first two modes or photon states correspond to the two transverse photons, and the third one to the longitudinal photon, or plasmon.

It is also clear, from this dispersion relation, that the electromagnetic field in a plasma is a kind of massive vector field where an equivalent mass vector can be defined, with components

$$m_\lambda = \frac{\hbar}{c_\lambda^2} \omega_p . \tag{9}$$

This definition extends the previous concept of the photon equivalent mass to include that of the longitudinal photon. We see that the effective mass of a plasmon is much larger than that of the transverse photons, because we usually have $c^2 \gg S_e^2$.

Using the Heisenberg equations for the creation and destruction operators, and the well known commutation relations between these operators, we can easily obtain

$$\frac{d}{dt} a^\dagger(k,\lambda) = i\omega_k(\lambda) a^\dagger(k,\lambda) \tag{10}$$

and a similar equation for $a(k,\lambda)$. These equations simply state that the field modes are quantum oscillators with frequencies $\omega_k(\lambda)$ determined by Eq. (8). In this sense, they are the exact quantum counterparts of the linearlized classical field equations (4) and (5).

Let us now turn to the nonlinear analysis. The ponderomotive force exerced by the transverse field on the plasma electrons can be included in the electron equations of motion. By using a nonlinear perturbative approach, we can write the total electron plasma density as the sum of three terms,

$$n = n_0 + \tilde{n} + n_2 , \tag{11}$$

where $\tilde{n}$ is the linear perturbation associated with the existence of electron plasma waves, and n_2 is the nonlinear contribution to the density perturbation due to the ponderomotive force. This last term is determined by the equation

$$\frac{\partial^2}{\partial t^2} n_2 + \omega_p^2 n_2 = \frac{e^2 n_0}{2m^2} \nabla^2 |A|^2 . \tag{12}$$

Here it should be noted that, for transverse wave packets moving without significant deformation accross the plasma with group velocity v_g, we can write

$$\nabla |A|^2 \simeq \frac{1}{v_g} \frac{\partial}{\partial t} |A|^2 . \tag{13}$$

It is well known that this nonlinear density perturbation is composed of two parts [3]. One is directly due to the gradient of the electromagnetic energy density which expels the electrons away from the region occupied by the transverse wave packets. The other is due to the electron density oscillations which are excited by the restoring electrostatic fields, and can stay much longer after the wave packet has gone. These electrostatic oscillations follow the transverse wave packets along the plasma as a kind of wake, and are usually called the wakefield.

In a first step, we can neglect the wakefield perturbation by dropping the second term on the left hand side of Eq. (12). This corresponds to concentrating on effects occurring on a time scale much faster than the period of the electron plasma oscillations: $(\partial^2/\partial t^2) \gg \omega_p^2$. Taking the approximate equation (13) into account, we can then obtain

$$n_2 \simeq \frac{\epsilon_0}{2m} k_p^2 |A|^2 , \tag{14}$$

where we have used $k_p^2 = (\omega_p / v_g)^2$. The same result could also be derived by using a more exact calculation [3].

The total charge density associated with this nonlinear density perturbation will then be

$$Q = -e n_2 = -\frac{e\epsilon_0}{2m} k_p^2 |A|^2 . \tag{15}$$

In more exact terms, this can be written as

$$Q = \int Q_k \frac{d\vec{k}}{(2\pi)^3} , \tag{16}$$

where

$$Q_k = -\frac{e\epsilon_0}{2m\omega_k} \frac{\omega_p^2}{v_k^2} \vec{A}_k^* \cdot \vec{A}_k \tag{17}$$

and $v_k = (\partial\omega_k / \partial k)$ is the group velocity associated with the wave vector $\vec{k}$.

However, the field quantization given by Eqs. (6) and (7) shows that we can replace the classical quantities $\vec{A}_k$ and $\vec{A}_k^*$ by the operators

$$\vec{A}_k \to \sum_{\lambda=1,2} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} a_k(\lambda) \vec{e}_k(\lambda) \tag{18}$$

and

$$\vec{A}_k^* \to \sum_{\lambda=1,2} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} a_k^\dagger(\lambda) \vec{e}_k^*(\lambda), \tag{19}$$

where $\vec{e}_k(\lambda)$ is the unit polarization vector. This means that, in the process of extending our quantization procedure to the nonlinear fields, the classical quantity $|A_k|^2$ appearing in Eq. (17) will be replaced by the following operator:

$$|A_k|^2 \to \sum_{\lambda=1,2} \frac{\hbar}{2\epsilon_0 \omega_k} a_k^\dagger a_k . \tag{20}$$

This means that we can define the photon charge operator as

$$Q_k(\lambda) = -\frac{e\hbar}{4m\omega_k}\frac{\omega_p^2}{v_k^2}a_k^\dagger(\lambda)a_k(\lambda). \tag{21}$$

Calculating the mean value of this operator for some quantum state $|\phi\rangle$, we can determine the mean value of the total photon charge $\langle Q\rangle = \langle\phi|Q|\phi\rangle$ as

$$\begin{aligned} Q &= \sum_{\lambda=1,2}\int \langle Q_k(\lambda)\rangle \frac{d\vec{k}}{(2\pi)^3} \\ &= -\sum_{\lambda=1,2}\int \frac{e\hbar}{4m\omega_k}\frac{\omega_p^2}{v_k^2}\langle a_k^\dagger(\lambda)a_k(\lambda)\rangle \frac{d\vec{k}}{(2\pi)^3}. \end{aligned} \tag{22}$$

This can also be written in a more appropriate form as

$$Q = \sum_{\lambda=1,2}\int q_k n_k(\lambda)\frac{d\vec{k}}{(2\pi)^3}. \tag{23}$$

In this expression, the quantity $n_k(\lambda) = \langle a_k^\dagger(\lambda)a_k(\lambda)\rangle$ is the photon occupation number, or the mean value of the usual number operator, and the quantity q_k is the photon charge, or the equivalent electric charge of a single photon:

$$q_k = -\frac{e\hbar}{4m\omega_k}\frac{\omega_p^2}{v_k^2}. \tag{24}$$

This quantity differs by a numerical factor of 2 from the quantity given in our previous work on the classical theory, which is due to a distinct (and eventually more appropriate) definition of the photon occupation number.

As an numerical example, let us consider the case of an optical photon moving in a dense plasma. For an electron plasma density of $n_0 = 10^{20}$ cm^{-3}, the photon effective mass is $m_{\lambda=1,2} \simeq 7\times 10^{-7}\times m$, where m is the electron mass, and it is independent of the photon frequency, as shown by Eq. (9). For a laser compressed plasma where the electron density can be as high as 10^{26} cm^{-3}, the photon effective mass will be 10^3 times larger.

Assuming that the photon frequency is ten times larger than the electron plasma frequency, $\omega_k \simeq 10\omega_p$, the equivalent charge will be $q_k \simeq -2\times 10^{-8}\times e$. This is much smaller (in absolute value) than the electron charge $-e$, but will become non-negligible for intense laser pulses containing a large number of photons inside a small volume. Furthermore, if ω_k approaches the cutoff frequency ω_p, the group velocity v_k will become much smaller than c and, according to definition (24), the value of the equivalent charge q_k will become much larger.

Finally, we should note that the wakefield perturbation associated with an electromagnetic wave packet could also be easily quantized, because it consists of electron plasma waves with a phase velocity nearly equal to the group velocity of the wave packet. This means that, in quantum terms, we can say that a group of transverse photons moving in a plasma is always followed by a wake of longitudinal photons, or plasmons. However, in this simple and suggestive quantum picture of the wakefield, we should not forget that individual plasmons have very low (group) velocities.

In conclusion, we have shown here that a straightforward quantization procedure can be used for a linearized electromagnetic field in a plasma, and that it naturally leads to the concept of the photon effective mass. This mass is different for transverse photons and for longitudinal photons (or plasmons), but it is always proportional to the electron plasma frequency. It was also shown that the field quantization in a plasma is very similar, but not identical, to the quantization of a massive vector field.

Using a nonlinear perturbative analysis we were able to extend this quantization procedure, in order to define a photon charge operator, and to establish the value of the photon equivalent charge. The single photon charge appears, in our quantum description of the photon field in a plasma, as the constant of proportionality between this charge operator and the well known photon number operator. The existence of such a charge is a result of the ponderomotive force (or radiation pressure) which pushes plasma electrons away from regions occupied by transverse photons.

The present analysis remains valid as long as the plasma medium can be described by a fluid model. This means that it will eventually break down for very high-energy photons, with a wavelength much smaller than the mean particle distance inside the plasma. However, even in that case, we can still find a nonzero photon charge and mass, as long as the interaction of the photon with the particles of the medium is formulated in terms of mean field concepts. This can be done, for instance, by relating the individual collisions of the high-energy photons with the individual plasma electrons to the plasma dielectric constant by using the optical theorem. This question will be addressed in a future work.

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